Secant Method For Solving Nonlinear Equations
Method Basics

Gary Schurman, MBE, CFA

December, 2022

In our series on the Newton-Raphson Method for solving non-linear equations we used the derivative of $f(x)$ to solve for the value of the independent variable x given the value of the dependent variable $f(x)$. In our series on the Secant Method we will use two points on the $f(x)$ line rather than the derivative of $f(x)$ to solve for x. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are given the following non-linear equation for $f(x)$...

$$f(x): \;\; 4\,x^2 = 27 \tag{1}$$

Question: Use the secant method to solve the equation above for the independent variable x.

Secant Method Equations

We are given the non-linear function $f(x)$ and the value of that function at x and are tasked with solving for the independent variable x. The form of the equation that we must solve is...

$$f(x) = c \;\text{ ...where... the variable } c \text{ is a known constant ...and... the independent variable } x \text{ is unknown} \tag{2}$$

To employ the secant method to solve for x we need to define the two points a and b such that...

$$f(a) < 0 \;\text{ ...and... }\; f(b) > 0 \;\text{ ...given that... }\; a < b \tag{3}$$

To ensure that the two points a and b exist, using Equation (2) above we will define our new function $g(x)$ to be...

$$g(x) = f(x) - c = 0 \;\text{ ...such that... }\; g(a) < 0 \;\text{ ...and... }\; g(b) > 0 \tag{4}$$

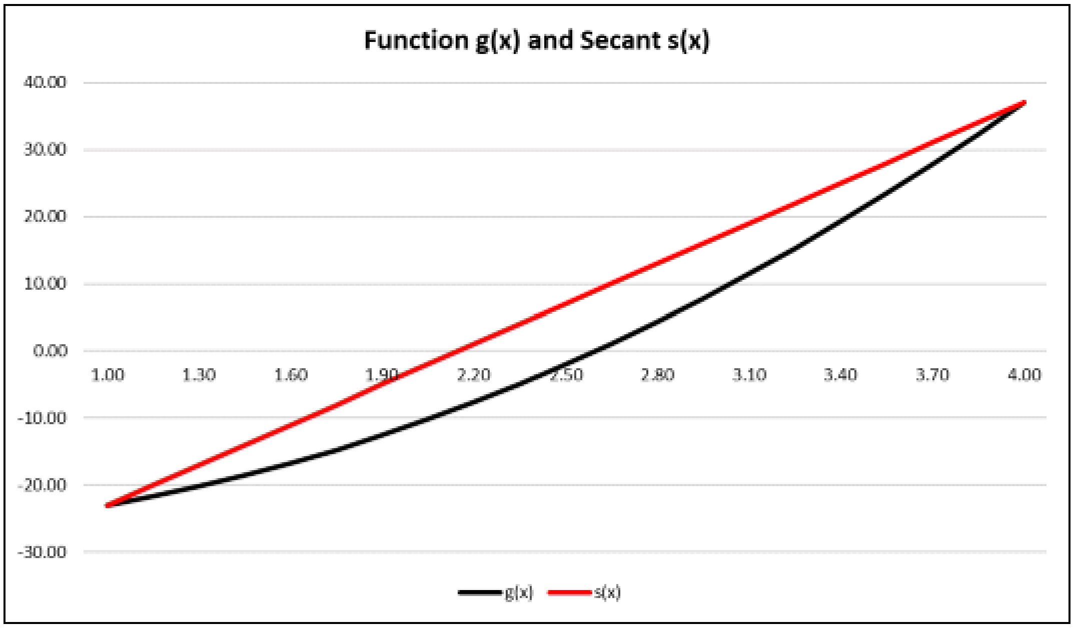
Using Equations (1) and (4) above the equation for $g(x)$ for our problem is...

$$g(x): \;\; 4\,x^2 - 27 = 0 \tag{5}$$

We will define the function $s(x)$ to be the secant line that connects points a and b. Using Equation (4) above the equation for the secant line is...

$$s(x) = m\,(x-a) + d \;\text{ ...where... }\; m = \text{slope} = \frac{g(b) - g(a)}{b - a} \;\text{ ...and... }\; d = \text{constant} = g(a) \tag{6}$$

The graph below shows the graph of our problem's function $g(x)$ and secant $s(x)$ at points $a = 1$ and $b = 4$. Note that at point a the value of the function $g(a) = s(a) = -23.00$ and at point b the value of the function $g(b) = s(b) = 37.00$



Note that in the graph above the actual value of the independent variable x is the point at which the function $g(x)$ crosses the x-axis, which looks to be somewhere between 2.50 and 2.80. Whereas the function $g(x)$ is non-linear and the value of x where $g(x) = 0$ is difficult to calculate, the function $s(x)$ is linear and the value of x where $s(x) = 0$ is easy to calculate. Therefore, to calculate the actual value of x we employ the following iteration...

Step	Action
1	Determine guess values for points a and b such that $g(a) < 0$ and $g(b) > 0$.
2	Calculate the secant equation $s(x)$ that connects points a and b.
3	Use the secant equation from Step 2 to solve for $\hat{a}$ such that $s(\hat{a}) = 0$.
4	Use the value of $\hat{a}$ from Step 3 and calculate the value of $g(\hat{a})$.
5	If $g(\hat{a}) \approx 0$ then stop, otherwise replace the value of a with $\hat{a}$ and go to Step 2.

Using Equations (5) and (6) the equation that we will use for $\hat{a}$ in Step 3 above is...

$$\text{if... } s(\hat{a}) = m\,(\hat{a} - a) + d = 0 \text{ ...then... } \hat{a} = a - \frac{d}{m} = a - f(a)\left(\frac{g(b) - g(a)}{b - a}\right)^{-1} \tag{7}$$

The Answer To Our Hypothetical Problem

Question: Use the secant method to solve the equation $f(x) :\ 4\,x^2 = 27$ for the independent variable x.

Our first task is to determine the equation for the function $g(x)$ such that...

$$\text{if... } g(x) = 0 \text{ ...then... } g(x) = 4\,x^2 - 27 = 0 \tag{8}$$

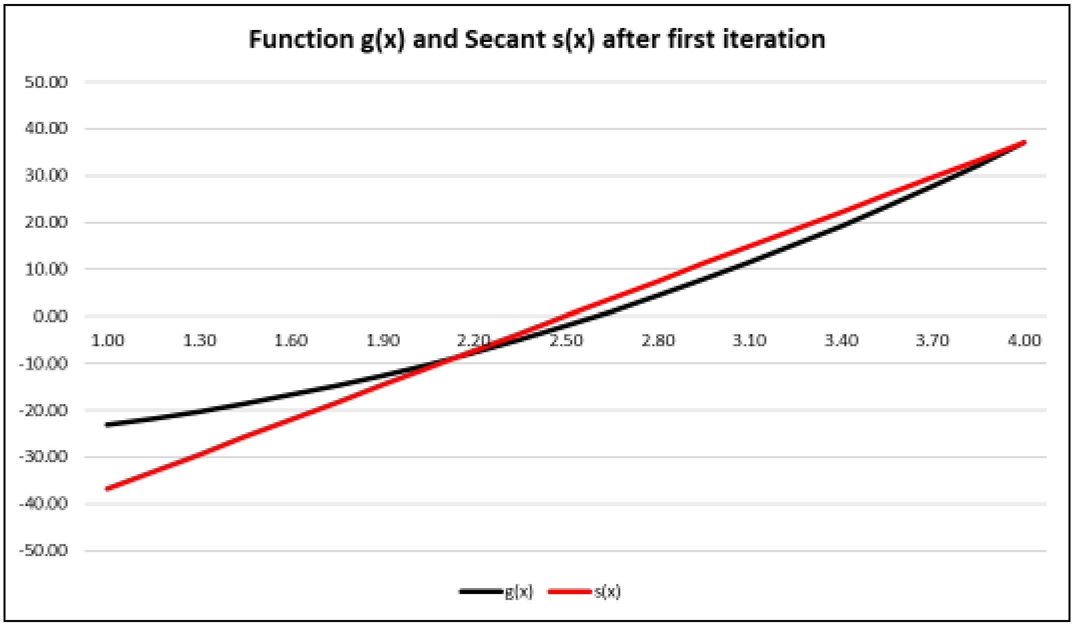
The next step is to choose values for the points a and b such that $g(a) < 0$ and $g(b) > 0$. If we choose the points $a = 1$ and $b = 4$ then using Equation (8) above...

$$g(a) = g(1) = -23.00 \text{ ...and... } g(b) = g(4) = 37.00 \text{ ...and... } f(a) = f(1) = 4 \times 1^2 - 27 = -23.00 \tag{9}$$

The next step is to plug the function values from Equation (9) above into iterated Equation (7) above and calculate the value of $\hat{a}$...

$$\hat{a} = 1.00 - (-23.00) \times \left(\frac{37.00 - (-23.00)}{4.00 - 1.00}\right)^{-1} = 2.15 \text{ ...such that... } g(\hat{a}) = 4 \times 2.15^2 - 27 = -8.51 \tag{10}$$

If the function $g(\hat{a})$ in Equation (10) above is approximately zero (or close enough) then we can stop. If $g(\hat{a})$ is materially different from zero then we replace the parameter a with $\hat{a}$ and perform the next iteration. The graph below presents our two functions after the first iteration where parameter a becomes $\hat{a}$, parameter b is unchanged, and the secant is recalculated...



Note that in the graph above the second iteration $g(\hat{a}) - f(\hat{a})$ is less than the first iteration $g(a) - f(a)$, which means that we a converging to the solution to our non-linear equation.

If we iterate Equation (7) above then the results of each iteration are...

Iteration	a	b	g(a)	g(b)	$\hat{a}$	$g(\hat{a})$
1	1.0000	4.0000	-23.0000	37.0000	2.1500	-8.5100
2	2.1500	4.0000	-8.5100	37.0000	2.4959	-2.0812
3	2.4959	4.0000	-2.0812	37.0000	2.5760	-0.4562
4	2.5760	4.0000	-0.4562	37.0000	2.5934	-0.0976
5	2.5934	4.0000	-0.0976	37.0000	2.5971	-0.0208
6	2.5971	4.0000	-0.0208	37.0000	2.5979	-0.0044
7	2.5979	4.0000	-0.0044	37.0000	2.5980	-0.0009
8	2.5980	4.0000	-0.0009	37.0000	2.5981	-0.0002
9	2.5981	4.0000	-0.0002	37.0000	2.5981	0.0000

Note that after each iteration we replace a with $\hat{a}$ and recalculate. If we determine that we want the solution to the problem to be accurate up to 4 decimal places then after iteration nine $g(\hat{a}) \approx 0$ such that the answer to our hypothetical problem is $x = 2.5981$. $(f(2.5981) = 4 \times 2.5981^2 = 27.00)$.

References

[1] Gary Schurman, *The Taylor Series Expansion,* November, 2017.